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THE CONTINUOUS PLANE MOTION OF A LIQUID BOUNDED BY TWO RIGHT LINES

BY HENRY C. WOLFF

THE following paper gives a discussion of the general equation transforming an infinite strip, of definite width, conformally upon the entire plane: the boundaries of the strip become any two non-intersecting lines, each extending to infinity in one direction. This general function, of which the hyperbolic sine,[*] the well known function of Helmholtz,[†] and the function recently discussed by Harris,[‡] are special cases, is expressed below by equation (10).

Let us consider a property of the special functions which led to the discovery of the general equation. From the equation

$$Z = \pi \sinh \frac{z}{2}$$

we have

$$\frac{dZ}{dz} = \zeta = \frac{\pi}{4}\left[e^{z/2} + e^{-z/2}\right].$$

Factoring the right-hand side as the difference of two perfect squares we obtain

$$\zeta = \frac{\pi}{4}\left[e^{z/4} - ie^{-z/4}\right]\left[e^{z/4} + ie^{-z/4}\right].$$

Let us write

$$\frac{dZ'}{dz} = \zeta' = \frac{\pi}{4}\left[e^{z/4} - ie^{-z/4}\right],$$

thus omitting the last factor of ζ.

Integrating we have

$$Z' = \pi\left[e^{z/4} + ie^{-z/4}\right]. \tag{1}$$

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[*] A. R. Forsythe, *Theory of Functions of a Complex Variable*, Cambridge, 1893, p. 503.

[†] H. von Helmholtz, *Wissenschaftliche Abhandlungen*, vol. 1, p. 154.

[‡] R. A. Harris, On Two-dimensional Fluid Motion through Spouts composed of Two Plane Walls, ANNALS OF MATHEMATICS, ser. 2, vol. 2, p. 73; 1902.

Separating the real and imaginary parts we have

$$X' = \pi\left[e^{x/4}\cos\frac{y}{4} + e^{-x/4}\sin\frac{y}{4}\right],$$

$$Y' = \pi\left[e^{x/4}\sin\frac{y}{4} + e^{-x/4}\cos\frac{y}{4}\right].$$

When $y = \pi$, let X and Y be represented by $X(\pi)$ and $Y(\pi)$ respectively. Then

$$X'(\pi) = \frac{\pi}{\sqrt{2}}\left[e^{x/4} + e^{-x/4}\right],$$

$$Y'(\pi) = \frac{\pi}{\sqrt{2}}\left[e^{x/4} + e^{-x/4}\right],$$

or

$$X'(\pi) = Y'(\pi),$$

the equation of a straight line in the Z'-plane. Here $X'(\pi)$ has its minimum value when $x = 0$.

When $y = -\pi$ we have

$$X'(-\pi) = \frac{\pi}{\sqrt{2}}\left[e^{x/4} - e^{-x/4}\right],$$

$$Y'(-\pi) = \frac{-\pi}{\sqrt{2}}\left[e^{x/4} - e^{-x/4}\right];$$

or

$$X'(-\pi) = -Y'(-\pi),$$

the equation of a straight line in the Z'-plane. Here $X'(-\pi)$ has neither a maximum nor a minimum value.

When $y = -3\pi$ we get

$$X'(-3\pi) = -\frac{\pi}{\sqrt{2}}\left[e^{x/4} + e^{-x/4}\right],$$

$$Y'(-3\pi) = -\frac{\pi}{\sqrt{2}}\left[e^{x/4} + e^{-x/4}\right];$$

or

$$X'(-3\pi) = Y'(-3\pi).$$

Here $X'(-3\pi)$ has its maximum value when $x = 0$.

Thus a strip of width 4π between the lines $y = \pi$ and $y = -3\pi$ on the z-plane is represented conformally upon the entire Z'-plane. The line $y = \pi$ is bent at the point $x = 0$, and the parts on either side are folded together forming one half line in the Z'-plane. The line $y = -3\pi$ is bent at the point $x = 0$, and the two parts are folded together forming the second half of the same line in the Z'-plane. The line $y = -\pi$ remains unbroken. Thus we see that the transformations

$$Z = \pi \sinh \frac{z}{2} \quad \text{and} \quad Z' = \pi[e^{z/4} + ie^{-z/4}]$$

are similar, but the strip transformed by the first equation is one-half as wide as, and coincides with the upper half of, the strip transformed by the second equation. In fact, with this object in view, the latter equation could have been built up directly from the former.

Let us now, considering the second variable factor of ζ, write

$$\frac{dZ''}{dz} = \zeta'' = \frac{\pi}{4}\left[e^{z/4} + ie^{-z/4}\right].$$

Integrating, we have

$$Z'' = \pi[e^{z/4} - ie^{-z/4}]. \tag{2}$$

Discussing this in the same way that we discussed Z' we find that it represents the strip between the lines $y = -\pi$ and $y = +3\pi$ in the z-plane conformally upon the entire Z''-plane.

Now let us note how the functions derived from these three ζ's transform the lines $y = \pi$ and $y = -\pi$ on the z-plane. That derived from ζ' leaves the line $y = -\pi$ unbent, while it bends the line $y = \pi$ at the point $x = 0$. That derived from ζ'' does just the opposite; it leaves the line $y = \pi$ unbent and bends the line $y = -\pi$ at the point $x = 0$. The function corresponding to ζ bends both of these lines at the point $x = 0$. We may then look upon the functions, corresponding to the two factors of ζ, as each doing a half of the transformation performed by the function from which they were derived. It is apparent that if the z in ζ' were first changed to $z + a$ the transformation represented by equation (1) would bend the line $y = \pi$ at the point $x = -a$ instead of at the point $x = 0$; and if in equation (2) the z were replaced by $z - a$ the transformation would bend the line $y = -\pi$ at the point $x = a$ instead of at $x = 0$.

Let us now investigate the transformation effected by a function corresponding to a ζ having as factors the ζ' and ζ'' with the z changed as indicated above. We should then have

$$\frac{dZ}{dz} = \zeta = \frac{\pi}{4}\left[e^{(z+a)/4} - ie^{-(z+a)/4}\right]\left[e^{(z-a)/4} + ie^{-(z-a)/4}\right],$$

or

$$\frac{dZ}{dz} = \zeta = \frac{\pi}{4}\left[e^{z/2} + e^{-z/2}\right] + \frac{\pi i}{4}\left[e^{a/2} - e^{-a/2}\right].$$

Integrating, we have

$$Z = \frac{\pi}{2}\left[e^{z/2} - e^{-z/2}\right] + \frac{\pi i K}{4}\, z, \tag{3}$$

where $K = e^{a/2} - e^{-a/2} = 2 \sinh \frac{a}{2}$.

Separating the real and imaginary parts, we have

$$X = \frac{\pi}{2}\left[e^{x/2} - e^{-x/2}\right]\cos\frac{y}{2} - \frac{\pi K}{4}\, y, \tag{4}$$

$$Y = \frac{\pi}{2}\left[e^{x/2} + e^{-x/2}\right]\sin\frac{y}{2} + \frac{\pi K}{4}\, x. \tag{5}$$

When $y = \pi$,

$$X(\pi) = -\frac{\pi^2 K}{4} \quad \text{and} \quad Y(\pi) = \frac{\pi}{2}\left[e^{x/2} + e^{-x/2}\right] + \frac{\pi K}{4}\, x.$$

When $y = -\pi$,

$$X(-\pi) = \frac{\pi^2 K}{4} \quad \text{and} \quad Y(-\pi) = -\frac{\pi}{2}\left[e^{x/2} + e^{-x/2}\right] + \frac{\pi K}{4}\, x.$$

Here $Y(\pi)$ has its minimum value when $x = -a$; and $Y(-\pi)$ has its maximum value when $x = a$; that is, the upper edge of the strip on the z-plane is bent at the point $x = -a$ and the lower edge at the point $x = a$, each becoming a vertical line in the new plane extending to infinity in one direction. The perpendicular distance between these walls* is $\pi^2 K/2$. Introducing a multiplier $4/(\pi K)$ in the right-hand side of equation (3) will keep the distance between the walls a constant equal to 2π. The over-lap of the two walls is then equal to

$$2a - 4\,\frac{e^a + 1}{e^a - 1},$$

which becomes zero when $a = 2.40 \cdots$. By a proper choice of the a the over-lap can be made any desired amount either positive or negative. The equation corresponding to the line $y = 0$ is

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* The two straight lines on the Z-plane corresponding to the lines $y = \pi$ and $y = -\pi$ on the z-plane will be termed the walls.

$$X = \frac{4}{K} \sinh \frac{Y}{2} \cdot$$

The drawing for the case $a = 4$ is given in figure 1, and for the case $a = 1$ in figure 2.

If we differentiate the Helmholtz equation, we obtain

$$\frac{dZ}{dz} = \zeta = 1 + e^z = (1 + ie^{z/2})(1 - ie^{z/2}).$$

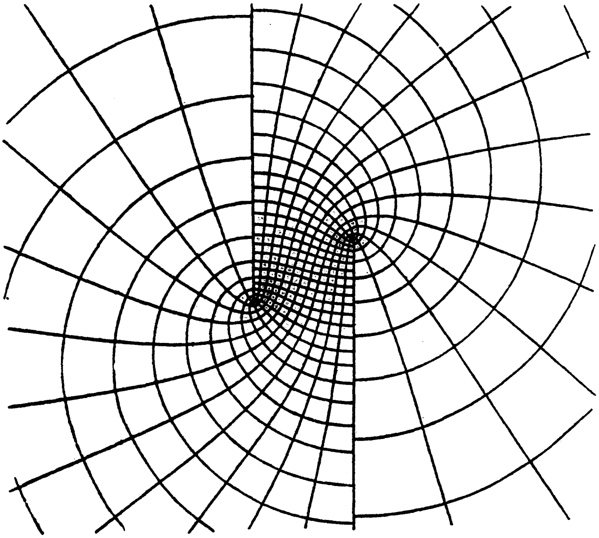

FIG. 1. ($\epsilon = \frac{1}{2}$, $a = 4$.)

Now if we use the first factor only in a new ζ we have

$$\frac{dZ'}{dz} = \zeta' = (1 + ie^{z/2}),$$

$$Z' = z + 2ie^{z/2}. \qquad (6)$$

Separating the real and imaginary parts, we have

$$X' = x - 2e^{x/2} \sin \frac{y}{2},$$

$$Y' = y + 2e^{x/2} \cos \frac{y}{2} \cdot$$

When $y = \pi$,

$$X'(\pi) = x - 2e^{x/2},$$

$$Y'(\pi) = \pi.$$

Here $X'(\pi)$ has its maximum value when $x = 0$.

When $y = -\pi$,

$$X'(-\pi) = x + 2e^{x/2},$$

$$Y'(-\pi) = -\pi.$$

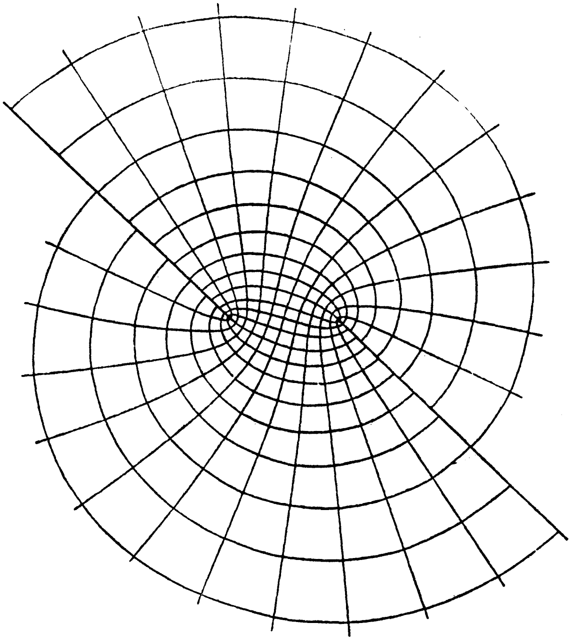

FIG. 2. ($\epsilon = \frac{1}{2}$, $a = 1$.)

Here $X'(-\pi)$ has neither a maximum nor a minimum value.

When $y = -3\pi$,

$$X'(-3\pi) = x - 2e^{x/2},$$

$$Y'(-3\pi) = -3\pi.$$

Here $X'(-3\pi)$ has its maximum value when $x = 0$. We see that here, as in the case of the transformation expressed by the hyperbolic sine, the function derived from the first factor of ζ is similar to the original Helmholtz transformation.

If we omit the first factor of ζ we have

$$\frac{dZ''}{dz} = \zeta'' = (1 - ie^{z/2}),$$

$$Z'' = z - 2ie^{z/2}. \qquad (7)$$

Here again, as in the case of the transformation expressed by the hyperbolic sine, the Z' and the Z'' functions may be considered as each performing one half of the transformation represented by the original function from which they were obtained.

If the z in ζ' were replaced by $z + a$, equation (4) would require that the line $y = \pi$ be bent at the point $x = -a$ instead of at the point $x = 0$. If in ζ'' the z were replaced by $z - a$ equation (7) would require that the line $y = -\pi$ be bent at the point $x = a$ instead of at the point $x = 0$. Let us now use ζ' and ζ'', with the z in the former changed to $z + a$, and the z in the latter changed to $z - a$, as factors of ζ. We then have

$$\frac{dZ}{dz} = \zeta = \left[1 + ie^{(z+a)/2}\right]\left[1 - ie^{(z-a)/2}\right] = 1 + e^z + iKe^{z/2};$$

$$Z = z + e^z + 2iKe^{z/2}. \qquad (8)$$

Separating the real and imaginary parts and substituting π and $-\pi$ for x, we get

$$X(\pi) = x - e^x - 2Ke^{x/2},$$

$$Y(\pi) = \pi,$$

$$X(-\pi) = x - e^x + 2Ke^{x/2},$$

and

$$Y(-\pi) = -\pi.$$

Here $X(\pi)$ has its maximum value when $x = -a$, and $X(-\pi)$ has its maximum value when $x = a$. The upper and lower edges of the strip on the z-plane are bent at the points $x = -a$ and $x = +a$, respectively, and are transformed into two horizontal walls extending to infinity in the same direction. One wall protrudes $2a - e^{-a} + e^a$ units beyond the other with a constant perpendicular distance, 2π, between them. By a proper choice of the a the protrusion can be made any desired amount. In figure 3 is given the drawing for $a = 1$.

By differentiating the Harris equation*

$$Z = \frac{\pi}{\sin \epsilon\pi}\left[\epsilon e^{(1-\epsilon)z} - (1-\epsilon)e^{-\epsilon z}\right], \tag{9}$$

where

$$0 \leqq \epsilon \leqq \tfrac{1}{2},$$

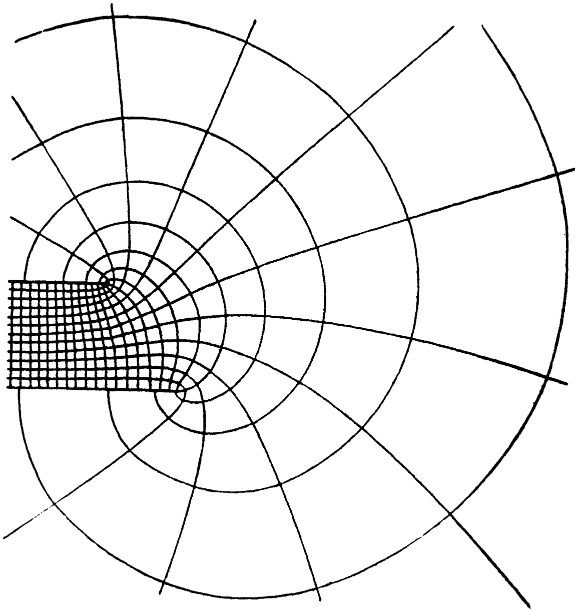

FIG. 3. ($\epsilon = 0$, $a = 1$.)

we obtain

$$\frac{dZ}{dz} = \frac{\pi\epsilon(1-\epsilon)}{\sin \epsilon\pi}\left[e^{(1-\epsilon)z} + e^{-\epsilon z}\right]$$

$$= \frac{\pi\epsilon(1-\epsilon)}{\sin \epsilon\pi}\left[e^{-\epsilon z/2} + ie^{(1-\epsilon)z/2}\right]\left[e^{-\epsilon z/2} - ie^{(1-\epsilon)z/2}\right].$$

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* Harris's equation reduces to this form by the substitutions $Z = \frac{Z'}{\pi}$, $z = z' + \log \frac{\epsilon}{1-\epsilon}$.

Omitting the last factor we have

$$\frac{dZ'}{dz} = \zeta' = \frac{\pi\epsilon(1-\epsilon)}{\sin\pi\epsilon}\left[e^{-\epsilon z/2} + ie^{(1-\epsilon)z/2}\right],$$

$$Z' = \frac{\pi}{\sin\epsilon\pi}\left[i\epsilon e^{(1-\epsilon)z/2} - (1-\epsilon)\,e^{-\epsilon z/2}\right].$$

By omitting the first factor of ζ we have

$$\frac{dZ''}{dz} = \zeta'' = \frac{\pi\epsilon(1-\epsilon)}{\sin\epsilon\pi}\left[-\,ie^{(1-\epsilon)z/2} + e^{-\epsilon z/2}\right],$$

$$Z'' = \frac{-\,2\pi}{\sin\epsilon\pi}\left[i\epsilon e^{(1-\epsilon)z/2} + (1-\epsilon)e^{-\epsilon z/2}\right].$$

These functions, Z' and Z'' effect transformations similar to the Z''s and Z'''s obtained from the hyperbolic sine and the Helmholtz function. Changing the z in ζ' to $z + a$, and in ζ'' to $z - a$, and multiplying, we get

$$\frac{dZ}{dz} = \frac{\pi\epsilon(1-\epsilon)}{\sin\epsilon\pi}\left[e^{-\epsilon(z+a)/2} + ie^{(1-\epsilon)(z+a)/2}\right]\left[e^{-\epsilon(z-a)/a} - ie^{(1-\epsilon)(z-a)/2}\right]$$

$$= \frac{\pi\epsilon(1-\epsilon)}{\sin\epsilon\pi}\left[e^{(1-\epsilon)z} + e^{-\epsilon z} + iKe^{(1-2\epsilon)z/2}\right],$$

$$Z = \frac{\pi}{\sin\epsilon\pi}\left[\epsilon e^{(1-\epsilon)z} - (1-\epsilon)e^{-\epsilon z} + \frac{2iK\epsilon(1-\epsilon)}{1-2\epsilon}\,e^{(1-2\epsilon)z/2}\right] + M + iN, \quad (10)$$

where
$$M = (1-2\epsilon)\left[\frac{\pi}{\sin\epsilon\pi} + 2 - \frac{K^2}{2}\right],$$

and
$$N = 2K\left[(1-2\epsilon) - \frac{\pi\epsilon(1-\epsilon)}{(1-2\epsilon)\sin\epsilon\pi}\right].$$

This is the general equation of transformation. Choosing the additive constant, $M + iN$, in the precise form selected above, gives symmetry with respect to the origin in the new plane and keeps the walls within the finite part of the plane in the limiting cases when $\epsilon = 1/2$ and $\epsilon = 0$.

When $a = 0$ we have as a special case the Harris transformation; when $\epsilon = 0$ we have equation (8) of which the Helmholtz equation is a special case when $a = 0$; when $\epsilon = 1/2$ we have equation (3) of which the hyperbolic sine is a special case when $a = 0$.

Separating the real and imaginary parts of equation (10) we have

$$Y = \frac{\pi}{\sin \epsilon\pi}\Big[\epsilon e^{(1-\epsilon)x}\sin(1-\epsilon)y + (1-\epsilon)e^{-\epsilon x}\sin\epsilon y + \frac{2K\epsilon(1-\epsilon)}{1-2\epsilon}e^{(1-2\epsilon)x/2}\cos\frac{1-2\epsilon}{2}y\Big] + N, \tag{11}$$

$$X = \frac{\pi}{\sin \epsilon\pi}\Big[\epsilon e^{(1-\epsilon)x}\cos(1-\epsilon)y - (1-\epsilon)e^{-\epsilon x}\cos\epsilon y - \frac{2K\epsilon(1-\epsilon)}{1-2\epsilon}e^{(1-2\epsilon)x/2}\sin\frac{1-2\epsilon}{2}y\Big] + M. \tag{12}$$

When $y = \pi$,

$$X(\pi) = \frac{-\pi}{\tan\epsilon\pi}\Big[\epsilon e^{(1-\epsilon)x} + (1-\epsilon)e^{-\epsilon x} + \frac{2K\epsilon(1-\epsilon)}{1-2\epsilon}e^{(1-2\epsilon)x/2}\Big] + M, \tag{13}$$

$$Y(\pi) = \pi\Big[\epsilon e^{(1-\epsilon)x} + (1-\epsilon)e^{-\epsilon x} + \frac{2K\epsilon(1-\epsilon)}{1-2\epsilon}e^{(1-2\epsilon)x/2}\Big] + N. \tag{14}$$

Eliminating x between equations (13) and (14) we have

$$Y(\pi) = -X(\pi)\tan\epsilon\pi + M\tan\epsilon\pi + N, \tag{15}$$

the equation of a straight line in the Z-plane.

Here $X(\pi)$ has its maximum when $x = -a$. When $x = -a$, let $X(\pi)$ and $Y(\pi)$ be represented by $X(\pi, -a)$ and $Y(\pi, -a)$ respectively.

$$X(\pi, -a) = \frac{-\pi e^{\epsilon a}}{\tan\epsilon\pi}\Big[\frac{1-\epsilon-\epsilon e^{-a}}{1-2\epsilon}\Big] + M, \tag{16}$$

$$Y(\pi, -a) = \pi e^{\epsilon a}\Big[\frac{1-\epsilon-\epsilon e^{-a}}{1-2\epsilon}\Big] + N. \tag{17}$$

When $y = -\pi$,

$$X(-\pi) = \frac{-\pi}{\tan\epsilon\pi}\Big[\epsilon e^{(1-\epsilon)x} + (1-\epsilon)e^{-\epsilon x} - \frac{2K\epsilon(1-\epsilon)}{1-2\epsilon}e^{(1-2\epsilon)x/2}\Big] + M, \tag{18}$$

$$Y(-\pi) = -\pi\Big[\epsilon e^{(1-\epsilon)x} + (1-\epsilon)e^{-\epsilon x} - \frac{2K\epsilon(1-\epsilon)}{1-2\epsilon}e^{(1-2\epsilon)x/2}\Big] + N. \tag{19}$$

Eliminating x between equations (18) and (19) we have

$$Y(-\pi) = X(-\pi)\tan\epsilon\pi - M\tan\epsilon\pi + N, \tag{20}$$

the equation of a straight line in the Z-plane.

Here $X\,(-\pi)$ has a maximum when $x = a$. When $x = a$, let $X(-\pi)$ and $Y(-\pi)$ be represented by $X(-\pi, a)$ and $Y(-\pi, a)$ respectively.

$$Y(-\pi,\, a) = \frac{-\pi e^{-\epsilon a}}{\tan \epsilon\pi}\left[\frac{1-\epsilon-\epsilon e^{a}}{1-2\epsilon}\right] + M, \tag{21}$$

$$Y(-\pi,\, a) = -\pi e^{-\epsilon a}\left[\frac{1-\epsilon-\epsilon e^{a}}{1-2\epsilon}\right] + N. \tag{22}$$

From equations (16), (17), (21), and (22) we find the length of the perpendicular let fall from the end of one wall upon the other to be

$$A = \frac{2\pi e^{\epsilon a}}{1-2\epsilon}\left[1 - e - \epsilon e^{-a}\right]\cos \epsilon\pi,$$

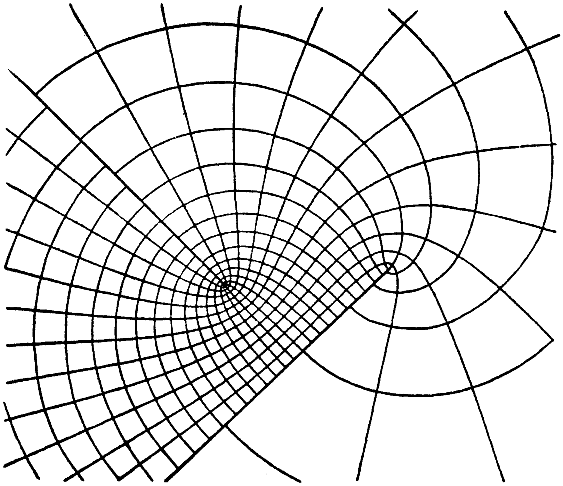

FIG. 4. ($\epsilon = \frac{1}{4}$, $a = 4$.)

and the protrusion of the second wall beyond the foot of this perpendicular to be

$$B = \frac{\pi}{(1-2\epsilon)\sin \epsilon\pi}\left[e^{\epsilon a}(1-\epsilon-\epsilon e^{-a})\cos 2\epsilon\pi - e^{-\epsilon a}(1-\epsilon-\epsilon e^{a})\right].$$

Then
$$\frac{B}{A} = \cot 2\epsilon\pi - \frac{e^{-2\epsilon a}}{\sin 2\epsilon\pi}\left[\frac{1-\epsilon-\epsilon e^{a}}{1-\epsilon-\epsilon e^{-a}}\right].$$

By properly choosing a this ratio may be made any desired amount for any value of ϵ. Thus we see that equation (10) represents conformally upon the entire plane a strip bounded by two parallel straight lines extending to infinity in both directions. The boundaries of the strip become any two non-intersecting right lines extending each to infinity in one direction.

Figure 4 represents the case $\epsilon = 1/4$ and $a = 4$.

In hydrodynamics this general transformation gives the continuous two-dimensional flow of a liquid in the Z-plane, having as boundaries two straight lines extending each to infinity in one direction. If the lines $y =$ constant are taken as stream lines within the infinite strip of width 2π on the z-plane, the corresponding curves on the new plane represent stream lines passing through the opening between the two walls. Similarly, if, in the z-plane, the line $y = \pi$ is considered a source, and the line $y = -\pi$ a sink, the lines $x =$ constant are transformed into stream lines passing from a rectilinear source to a rectilinear sink.

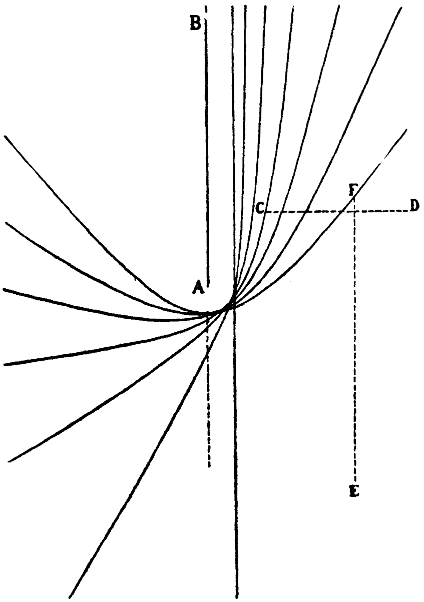

FIG. 5.

In figure 5, AB represents one wall held stationary while the second wall is allowed to move about in the plane as a and ϵ vary. . CD is a particular position of the second wall when at right angles to the first, i. e., $\epsilon = 1/4$. As a varies, or as the wall CD moves parallel to itself, the point C will describe a curve. Similar curves are drawn also for $\epsilon = 0$, $1/12$, $1/6$, $1/3$, and $1/2$.

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